A remark on amoebas in higher codimensions

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Abstract

It is shown that tube sets over amoebas of algebraic varieties of dimension q in $\mathbb{C}_*^n$ (and, more generally, of almost periodic holomorphic chains in $\mathbb{C}^n$) are q-pseudoconcave in the sense of Rothstein. This is a direct consequence of a representation of such sets as supports of positive closed currents.

1 Introduction

Let V be an algebraic variety in $\mathbb{C}_*^n = (\mathbb{C} \setminus 0)^n$. Its image $\mathcal{A}_V = \mathrm{Log}\, V$ under the mapping $\mathrm{Log}\,(z_1, \ldots, z_n) = (\log |z_1|, \ldots, \log |z_n|)$ is called the *amoeba* of A. The notion was introduced in [9] and has found numerous applications in complex analysis and algebraic geometry, see the survey [11].

The amoeba of V is a closed set with non-empty complement $\mathcal{A}_V^c = \mathbb{R}^n \setminus \mathcal{A}_V$. If V is of codimension 1, then each component of $\mathcal{A}_V^c$ is convex because $\mathrm{Log}^{-1}(\mathcal{A}_V^c)$ is the intersection of a family of domains of holomorphy. This is no longer true for varieties of higher codimension; nevertheless, some rudiments of convexity do take place. As shown by Henriques [10], if $\mathrm{codim}\, V = k$, then $\mathcal{A}_V^c$ is $(k-1)$-convex, a notion defined in terms of homology groups for sections by k-dimensional affine subspaces. A local result, due to Mikhalkin [11], states that $\mathcal{A}_V$ has no *supporting* k*-cap*, i.e., a ball B in a k-dimensional plane such that $\mathcal{A}_V \cap B$ is nonempty and compact, while $\mathcal{A}_V \cap (B + \epsilon v) = \emptyset$ for some $v \in \mathbb{R}^n$ and all sufficiently small $\epsilon > 0$.

The notion of amoeba was adapted by Favorov [3] to zero sets of holomorphic almost periodic functions in a tube domain as "shadows" cast by the zero sets to the base of the domain; a precise definition is given in Section 4. In [2], Henriques' result was extended to amoebas of zero sets of so-called regular holomorphic almost periodic mappings. This was done by a reduction to the case considered in [10] where the proof was given by methods of algebraic geometry.

In this note, we propose a different approach to convexity properties of amoebas in higher codimensions. It is purely analytical and works equally well for both algebraic and almost periodic situations. Moreover, we get our (pseudo)convexity results as a by-product of a representation of an amoeba as the support of a certain natural measure determined by the "density" of the zero set.

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Let us start with a hypersurface case. When $V = \{P(z) = 0\} \subset \mathbb{C}_*^n$ is defined by a Laurent polynomial P, the function

$$N_P(y) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} \log |P(e^{y_1+i\theta_1}, \dots, e^{y_n+i\theta_n})| \, d\theta,$$

known in tropical mathematics community as the *Ronkin function*, is convex in $\mathbb{R}^n$ and linear precisely on each connected component of $\mathcal{A}_V^c$. This means that the support of the current $dd^c N_P(\operatorname{Im} z)$ equals $T_{\mathcal{A}_V} = \mathbb{R}^n + i\mathcal{A}_V$, the tube set in $\mathbb{C}^n$ with base $\mathcal{A}_V$. Since the complement to the support of a positive closed current of bidegree $(1,1)$ is pseudoconvex (being a domain of existence for a pluriharmonic function), this implies pseudoconvexity of $T_{\mathcal{A}_V^c}$ and thus convexity of every component of $\mathcal{A}_V^c$. Of course, the function N_P gives much more than simply generating the amoeba (see, for example, [18], [8], [12]).

The same reasoning applies to amoebas of holomorphic almost periodic functions f with Ronkin's function N_P replaced by the mean value $\mathcal{M}_f(y)$ of $\log |f|$ over the real planes $\{x + iy : \, x \in \mathbb{R}^n\}$, $y \in \mathbb{R}^n$.

What we will do in the case of codimension $k > 1$, is presenting $T_{\mathcal{A}_V}$ as the support of a closed positive current of bidegree (k, k) (namely, a mean value current for the variety or, more generally, for a holomorphic chain) and then using a theorem on $(n - k)$-pseudoconcavity, in the sense of Rothstein, of supports of such currents due to Fornaess and Sibony [7]. In addition, we show that for a closed set $\Gamma \subset \mathbb{R}^n$, Rothstein's $(n - k)$-pseudoconcavity of T_Γ implies the absence of k-supporting caps of the set Γ (Proposition 2.2).

We obtain our main result, Theorem 4.1, for arbitrary almost periodic holomorphic chains, which is a larger class than zero sets of regular almost periodic holomorphic mappings, and the situation with algebraic varieties (Corollary 4.2) is a direct consequence. The existence of the mean value currents was established in [4], so here we just combine it together with the theorem on supports of positive closed currents. In this sense, this note is just a simple illustration of how useful the mean value currents are.

2 Rothstein's q-pseudoconvexity

We will use the following notion of q-pseudoconvexity, due to W. Rothstein [19], see also [14]. Given $0 < q < n$ and $\alpha, \beta \in (0, 1)$, the set

$$H = \{(z, w) \in \mathbb{C}^{n-q} \times \mathbb{C}^q : \, \|z\|_\infty < 1, \, \|w\|_\infty < \alpha \text{ or } \beta < \|z\|_\infty < 1, \, \|w\|_\infty < 1\}$$

is called an $(n - q, q)$-Hartogs figure; here $\|z\|_\infty = \max_j |z_j|$. Note that its convex hull $\hat{H}$ is the unit polydisc in $\mathbb{C}^n$. An open subset Ω of a complex n-dimensional manifold M is said to be *q-pseudoconvex* in M if for any $(n - q, q)$-Hartogs figure H and a biholomorphic map $\Phi : \hat{H} \to M$, the condition $\Phi(H) \subset \Omega$ implies $\Phi(\hat{H}) \subset \Omega$. If this is the case, we will also say that $M \setminus \Omega$ is *q-pseudoconcave* in M.

Loosely speaking, the q-pseudoconvexity is the Kontinuitätssatz with respect to $(n - q)$-polydiscs; usual pseudoconvexity is equivalent to $(n - 1)$-pseudoconvexity.

Theorem 2.1 ([7], Cor. 2.6) *The support of a positive closed current of bidimension (q,q) on a complex manifold M is q-pseudoconcave in M.*

It is easy to see that for tube sets, $(n-k)$-pseudoconcavity implies absence of k-caps in the sense of Mikhalkin.

Proposition 2.2 *Let Γ be a closed subset of a convex open set $D \subset \mathbb{R}^n$. If the tube set $T_\Gamma = \mathbb{R}^n + i\Gamma$ is $(n-k)$-pseudoconcave in the tube domain $T_D = \mathbb{R}^n + iD$, then Γ has no k-supporting caps.*

Proof: Assume Γ has a k-supporting cap B. We assume that the vector v in the definition of the cap is orthogonal to B (in the general case, one gets an image of a Hartogs figure under a non-degenerate linear transform). Choose coordinates in $\mathbb{R}^n$ such that

$$B = \{(y', y'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \ \|y'\|_\infty < 1, \ y'' = 0\},$$

$$\{\beta < \|y'\|_\infty < 1, \ \|y''\|_\infty < 1\} \subset D \setminus \Gamma, \quad \beta \in (0,1),$$

and $B + \epsilon v \subset D \setminus \Gamma$ for all $\epsilon \in (0,1)$, where $v = (0, v'')$, $\|v''\|_\infty = 1$. Since D is open, $\{y : \ \|y'\|_\infty < 1, \ \|y'' - \frac{1}{2}v''\|_\infty < \alpha\} \subset D \setminus \Gamma$ for some $\alpha \in (0,1)$. Therefore, the $\frac{i}{2}v''$-shift of the corresponding $(k, n-k)$-Hartogs figure H is a subset of the tube set $T_D \setminus T_\Gamma$. Since B is a subset of the shifted polydisc $\hat{H} + \frac{i}{2}v''$ and $B \cap \Gamma \neq \emptyset$, the set T_Γ is not $(n-k)$-pseudoconcave. $\square$

3 Almost periodic holomorphic chains

Here we recall some facts from Ronkin's theory of holomorphic almost periodic mappings and currents; for details, see [15], [17], [4], [5], and the survey [6].

Let $\mathcal{T}_t$ denote the translation operator on $\mathbb{R}^n$ by $t \in \mathbb{R}^n$, then for any function f on $\mathbb{R}^n$, $(\mathcal{T}_t^* f)(x) = f(\mathcal{T}_t x) = f(x+t)$.

A continuous mapping f from $\mathbb{R}^n$ to a metric space X is called *almost periodic* if the set $\{\mathcal{T}_t^* f\}_{t \in \mathbb{R}^n}$ is relatively compact in $C(\mathbb{R}^n, X)$ with respect to the topology of uniform convergence on $\mathbb{R}^n$. The collection of all almost periodic mappings from $\mathbb{R}^n$ to X will be denoted by $\mathrm{AP}(\mathbb{R}^n, X)$.

As is known from classical theory of almost periodic functions, any function $f \in \mathrm{AP}(\mathbb{R}^n, \mathbb{C})$ has its mean value $\mathcal{M}_f$ over $\mathbb{R}^n$,

$$\mathcal{M}_f = \lim_{s \to \infty} (2s)^{-n} \int_{\Pi_s} f \, dm_n,$$

where $\Pi_s = \{x \in \mathbb{R}^n : \ \|x\|_\infty < s\}$ and m_n is the Lebesgue measure in $\mathbb{R}^n$.

Let D be a convex domain in $\mathbb{R}^n$, $T_D = \mathbb{R}^n + iD$. A continuous mapping $f : T_D \to X$ is called *almost periodic on* T_D if $\{\mathcal{T}_t^* f\}_{t \in \mathbb{R}^n}$ is a relatively compact subset of $C(T_D, X)$ with respect to the topology of uniform convergence on each tube subdomain $T_{D'}$, $D' \Subset D$. The collection of all almost periodic mappings from T_D to X will be denoted by $\mathrm{AP}(T_D, X)$.

The set $\mathrm{AP}(T_D,\mathbb{C})$ can be defined equivalently as the closure (with respect to the topology of uniform convergence on each tube subdomain $T_{D'}$, $D' \Subset D$) of the collection of all exponential sums with complex coefficients and pure imaginary exponents (frequencies). The mean value of $f \in \mathrm{AP}(T_D,\mathbb{C})$ is a continuous function of $\mathrm{Im}\, z$. The collection of all holomorphic mappings $f \in \mathrm{AP}(T_D,\mathbb{C}^k)$ will be denoted by $\mathrm{HAP}(T_D,\mathbb{C}^k)$. In particular, any mapping from $\mathbb{C}^n$ to $\mathbb{C}^k$, whose components are exponential sums with pure imaginary frequencies, belongs to $\mathrm{HAP}(\mathbb{C}^n,\mathbb{C}^k)$.

The notion of almost periodicity can be extended to distributions. For example, a measure μ on T_D is called almost periodic if $\phi(t) = \int (\mathcal{T}_t)_* \phi \, d\mu \in \mathrm{AP}(\mathbb{R}^n,\mathbb{C})$ for every continuous function ϕ with compact support in T_D. Furthermore, it can be extended to holomorphic chains as follows.

Let $Z = \sum_j c_j V_j$ be a holomorphic chain on $\Omega \subset \mathbb{C}^n$ supported by an analytic variety $|Z| = \cup_j V_j$ of pure dimension q. Its integration current $[Z]$ acts on test forms ϕ of bidegree (q,q) with compact support in Ω (shortly, $\phi \in \mathcal{D}_{q,q}(\Omega)$) as

$$([Z],\phi) = \int_{Reg|Z|} \gamma_Z \phi = \sum_j c_j \int_{Reg\, V_j} \phi,$$

where the function γ_Z takes constant positive integer values on the connected components of $Reg|Z|$. The q-dimensional volume of Z in a Borel set $\Omega_0 \subset \Omega$ is

$$Vol_Z(\Omega_0) = \int_{\Omega_0 \cap Reg|Z|} \gamma_Z \beta_q$$

(the mass of the trace measure of $[Z]$ in Ω_0). If f is a holomorphic mapping on Ω such that $|Z| = f^{-1}(0)$ and $\gamma_Z(z)$ equals the multiplicity of f at z, the chain will be denoted by Z_f.

A q-dimensional holomorphic chain Z on T_D is called an *almost periodic holomorphic chain* if $(\mathcal{T}_t^*[Z],\phi) \in AP(T_D,\mathbb{C})$ for any test form $\phi \in \mathcal{D}_{q,q}(T_D)$. Here $\mathcal{T}_t^* S = \sum \alpha_{IJ}(z+t)\, dz^I \wedge d\bar{z}^J$ is the pullback of the current $S = \sum \alpha_{IJ}(z)\, dz_I \wedge d\bar{z}_J$.

For any $f \in \mathrm{HAP}(T_D,\mathbb{C})$, the chain (divisor) Z_f is always almost periodic; on the other hand, there exist almost periodic divisors (starting already from dimension $n=1$) that are not divisors of any holomorphic almost periodic function; when $n>1$, even a periodic divisor need not be the divisor of a periodic holomorphic function [16]. The situation with higher dimensional mappings is even worse, since the chain Z_f generated by $f \in \mathrm{HAP}(T_D,\mathbb{C}^k))$, $k>1$, need not be almost periodic [4]. It is however so if the mapping f is *regular*, that is, if $\mathrm{codim}\,|Z_g| = k$ or $|Z_g| = \emptyset$ for every mapping g from the closure of the set $\{\mathcal{T}_t^* f\}_{t\in\mathbb{R}^m}$ [4], [5]. A sufficient regularity condition [15] shows that such mappings are generic.

Now we can turn to construction of the current that plays central role in our considerations, the details can be found in [5]. Let Z be an almost periodic holomorphic chain of dimension q. For any test form $\phi \in \mathcal{D}_{q,q}(T_D)$, the mean value $\mathcal{M}_{\phi_Z}$ of the function $\phi_Z(t) := (\mathcal{T}_t^*[Z],\phi) \in \mathrm{AP}(\mathbb{R}^n,\mathbb{C})$ defines the *mean value current* $\mathcal{M}_Z$ of Z by the relation

$$(\mathcal{M}_Z,\phi) = \mathcal{M}_{\phi_Z}.$$

The current is closed and positive. Since $\mathcal{M}_Z$ is translation invariant with respect to x, its coefficients have the form $\mathcal{M}_{IJ} = m_n \otimes \mathcal{M}'_{IJ}$, where $\mathcal{M}'_{IJ}$ are Borel measures in D. In addition, if $\psi = \sum \psi_{IJ} dz_I \wedge d\bar{z}_J$ is a form with coefficients $\psi_{IJ} \in \mathcal{D}(D)$ and χ_s is the characteristic function of the cube Π_s, then there exists the limit

$$\lim_{s\to\infty} (2s)^{-n}([Z], \chi_s\psi) = (\mathcal{M}'_Z, \psi'),$$

where $\mathcal{M}'_Z = \sum \mathcal{M}'_{IJ} dy_I \wedge dy_J$ and $\psi' = \sum \psi_{IJ} dy_I \wedge dy_J$.

The trace measure $\mu_Z = \mathcal{M}_Z \wedge \beta_q$ can also be written as $\mu_Z = m_n \otimes \mu'_Z$, where μ'_Z is a positive Borel measure on D. The following result shows that it can be viewed as a density of the chain Z along $\mathbb{R}^n$.

Theorem 3.1 ([4], [5]) *Let Z be an almost periodic holomorphic chain in a tube domain T_D. For any open set $G \Subset D$ such that $\mu'_Z(\partial G) = 0$, one has*

$$\lim_{s\to\infty} (2s)^{-n} Vol_Z(\Pi_s + iG) = \mu'_Z(G);$$

in addition, $\mu'_Z(G) = 0$ if and only if $|Z| \cap T_G = \emptyset$.

Remark 3.2 For $Z = Z_f$ with regular $f \in \mathrm{HAP}(T_D, \mathbb{C}^k)$, Theorem 3.1 was proved in [15] (for $k = n$) and [13] ($k < n$), without using the notion of almost periodic chain. The current $\mathcal{M}_{Z_f}$ can be constructed as follows. The coefficients a_{IJ} of the current $\log|f|(dd^c \log|f|)^{k-1}$ are locally integrable functions on T_D, almost periodic in the sense of distributions: $(\mathcal{T}_t^* a_{IJ}, \phi) \in \mathrm{AP}(T_D, \mathbb{C})$ for any test function $\phi \in \mathcal{D}(T_D)$. Therefore, they possess their mean values $A_{IJ} = \mathcal{M}_{a_{IJ}}$, and the current $\mathcal{M}_{Z_f} = dd^c(\sum A_{IJ} dz_I \wedge d\bar{z}_J)$.

4 Amoebas

Following [3], if Z is an almost periodic holomorphic chain in T_D, then its *amoeba* $\mathcal{A}_Z$ is the closure of the projection of $|Z|$ to D:

$$\mathcal{A}_Z = \overline{\mathrm{Im}\,|Z|},$$

where the map $\mathrm{Im} : \mathbb{C}^n \to \mathbb{R}^n$ is defined by $\mathrm{Im}\,(z_1, \dots, z_n) = (\mathrm{Im}\, z_1, \dots, \mathrm{Im}\, z_n)$. When $Z = Z_f$ for a regular mapping $f \in \mathrm{HAP}(T_D, \mathbb{C}^p)$, we write simply $\mathcal{A}_f$.

Our convexity result is stated in terms of the tube set $T_{\mathcal{A}_Z} = \mathbb{R}^n + i\mathcal{A}_Z$.

Theorem 4.1 *If Z is an almost periodic holomorphic chain of dimension q in a tube domain $T_D \subseteq \mathbb{C}^n$, then $T_{\mathcal{A}_Z} = \mathrm{supp}\,\mathcal{M}_Z$, where $\mathcal{M}_Z$ is the mean value current of the chain Z. Therefore, $T_{\mathcal{A}_Z}$ is q-pseudoconcave in T_D. In particular, for any regular mapping $f \in \mathrm{HAP}(T_D, \mathbb{C}^k)$, the set $T_{\mathcal{A}_f}$ is $(n-k)$-pseudoconcave.*

Proof: By Theorem 3.1, $\mathcal{A}_Z = \mathrm{supp}\,\mu'_Z$, which can be rewritten as

$$T_{\mathcal{A}_Z} = \mathrm{supp}\, m_n \otimes \mu'_Z = \mathrm{supp}\,\mathcal{M}_Z.$$

Since the current $\mathcal{M}_Z$ is positive and closed, Theorem 2.1 implies the corresponding pseudoconcavity. $\square$

This covers the algebraic case as well by means of the map $\mathrm{E} : \mathbb{C}^n \to \mathbb{C}_*^n$, $\mathrm{E}(z_1, \dots, z_n) = (e^{-iz_1}, \dots, e^{-iz_n})$. For a Laurent polynomial P, the exponential sum E^*P is periodic in $T_{\mathbb{R}^n}$, and its mean value $\mathcal{M}_{\log|E^*P|}$ coincides with the Ronkin function N_P. Furthermore, given an algebraic variety $V \subset \mathbb{C}_*^n$, its pullback E^*V is almost periodic (actually, periodic) in $\mathbb{C}^n$ and $\mathcal{A}_{\mathrm{E}^*V} = \mathcal{A}_V$, which gives

Corollary 4.2 *The set $T_{\mathcal{A}_V^c}$ for an algebraic variety $V \subset \mathbb{C}_*^n$ of pure codimension k is $(n-k)$-pseudoconvex.*

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